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THE EFFICIENCY OF AN ESTIMATOR. APPLICATION

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Abstract *In this paper we present a definition of an efficient estimator for an unknown parameter and an example.*

The optimal estimator is called the minimum variance unbiased estimator. This type of estimator will be our choice for the optimum or best estimator.

The entire article consists my own point of view concerning the estimation theory in general and the efficient estimators in particular.

Key words:
statistical estimation, efficient estimator, random variable, relative efficiency, absolute correct estimator

JEL Codes:
C13-Estimation

1. INTRODUCTION

Let X be a random variable defined on a probability space (Ω, K, P) and its probability density function (or probability function) $f(x;\theta)$ which depends on a parameter θ with values in a specified parameter space $D_\theta \subseteq \mathrm{R}$. The parameter space D_θ is an open interval or region of an Euclidean space. To each value of θ, $\theta \in D_\theta$, it corresponds one member of the family which will be denoted by the symbol $\{f(x;\theta);\theta \in D_\theta\}$. Any member of this family of probability density functions will be denoted by the symbol $f(x;\theta);\theta \in D_\theta$.

Let $S_n(X) = (X_1, X_2, ..., X_n)$ denote a random sample of size n from a distribution that has a density (probability) function $f(x;\theta);\theta \in D_\theta$, where θ is an unknown (real) parameter. If θ_0 is the true value of θ then, obviously, $\theta_0 \in D_\theta$.

A first very important problem of statistical estimation is that of defining a statistic (function) $\hat{\theta} = g_n(X_1, X_2, \ldots, X_n)$, so that if $x_1, x_2, ..., x_n$ are the observed experimental values of $X_1, X_2, \ldots, X_n$ then the number $\hat{\theta}_0 = g_n(x_1, x_2, \ldots, x_n)$

represents an estimate of θ, that is, we have $g_n : \mathrm{R}^n \to \mathrm{R}$, $\mathrm{R}^n = \mathrm{R} \times \mathrm{R} \times \ldots \times \mathrm{R}$ where R^n is the sample space. In this case the statistic (that is a function on the sample variables) $g_n(X_1, X_2, \ldots, X_n)$ represents an estimator for the unknown parameter θ.

We can construct an infinite number of such statistics (functions) but, from them, we must select such an estimator which can be qualified as a good estimation point of θ. This can be achieved in one way by selecting $\hat{\theta} = g_n(X_1, X_2, \ldots, X_n)$ in such a way that not only $\hat{\theta}$ is an unbiased estimator of θ but also the variance of $\hat{\theta}$ is as small as possible. We do this because the variance of $\hat{\theta}$ is a measure of the intensity of the probability concentration for $\hat{\theta}$ in the neighborhood of the point $\theta = E(\hat{\theta})$.

In conclusion, in order to make a feasible choice of the estimator we must have some measures of this estimator quality, namely: bias, consistency, efficiency, and sufficiency.

Another very important problem in the statistical estimation concerns the estimation of the random variables. Consider two random variables X and Y. Suppose that only X can be observed. If X and Y are correlated, we may expect that knowing the value of X allows us to make some inference about the value of the unobserved variable Y. In this case, an interesting problem is issued: estimating one random variable with another (or one random vector with another).

If we consider any function $\hat{X}=g(X)$ on X, then this is called an estimator for Y. A desirable property of any estimator X of Y would be $E(\hat{X})=Y$,that is, $\hat{X}$ is an unbiased estimator for Y. Then, the error in the estimate can be written as $e=\hat{X}-E(\hat{X})$. This error is a random variable. We cannot minimize the error directly but we must choose some arbitrary function of e. to minimize. An intuitive and appropriate choice is the average mean square error of the components of e. In other words, we choose to minimize the diagonal terms of the following matrix

$$K_e= E[(\hat{X}-E(\hat{X}))(\hat{X}-E(\hat{X}))^T]$$

where K_e is just the covariance matrix of the estimator $\hat{X}$. Therefore, the diagonal terms of this matrix are the variances of the estimator's components. The estimator in this case is called the minimum variance unbiased estimator. This type of estimator will be our choice for the optimum or best estimator.

2. THE EFFICIENCY OF AN ESTIMATOR

Definition 1 The efficiency of the estimator $\hat{\theta}_n$ for the unknown parameter θ is measured by

$$E[(\hat{\theta}_n - \theta)^2] .$$

Definition 2 An estimator $\hat{\theta}_n$ for the unknown parameter θ having the minimum variance is called an efficient estimator.

Theorem 1 Let $S_n(X)=(X_1,X_2,..., X_n)$ be the bernoullian random sample of X characteristic that has a normal distribution with: m - unknown parameter, σ^2 - known parameter. Then, the statistic $\overline{X}$ is more efficiently than the statistic $\tilde{X}$, where

$$\overline{X}=\frac{1}{n}\sum_{k=1}^{n}X_k,$$

$$\tilde{X}=\frac{2}{n(n+1)}\sum_{k=1}^{n}kX_k.$$

Proof In case of a bernoullian sample of characteristic that has a normal distribution, we have the values:

$$E(X)=E(X_k)=m,\ k=\overline{1,n},$$
$$Var(X)=Var(X_k)=\sigma^2,\ k=\overline{1,n}.$$

If we consider our hypothesis: m – unknown parameter, σ^2 - known parameter, we obtain for the statistic $\overline{X}$ the relations:

$$E(\overline{X}) = m,$$
$$Var(\overline{X}) = \frac{\sigma^2}{n} \to 0, if\ n \to \infty.$$

Therefore the statistic $\overline{X}$ represents an absolute correct estimator of the unknow parameter m.

For the statistic $\tilde{X}$ we obtain

$$E(\tilde{X})=\frac{2}{n(n+1)}\sum_{k=1}^{n}k\underbrace{E(X_k)}_{=m}=$$
$$=\frac{2m}{n(n+1)}\underbrace{\sum_{k=1}^{n}k}_{=\frac{n(n+1)}{2}}=$$
$$=m,$$

respectively

$$Var(\tilde{X})=$$
$$=\frac{2^2}{n^2(n+1)^2}\sum_{k=1}^{n}k^2\underbrace{Var(X_k)}_{=\sigma^2}=$$
$$=\frac{4\sigma^2}{n^2(n+1)^2}\underbrace{\sum_{k=1}^{n}k^2}_{=\frac{n(n+1)(2n+1)}{6}}=$$
$$=\frac{\sigma^2}{n}\frac{4n+2}{3n+3},$$

that is

$$E(\tilde{X}) = m,$$
$$Var(\tilde{X}) = \frac{\sigma^2}{n}\frac{4n+2}{3n+3} \to 0,\ n \to \infty$$

Therefore , also the statistic $\tilde{X}$ represents an absolute correct estimator of the unknown parameter m.

We get

$$Var(\overline{X})=\frac{\sigma^2}{n}<Var(\tilde{X})=$$
$$=\frac{\sigma^2}{n}\frac{4n+2}{3n+3}=\frac{\sigma^2}{n}(1+\frac{n-1}{3n+3}),$$

that is

$$Var(\overline{X}) < Var(\tilde{X}),\ if\ n > 1.$$

<u>Conclusion</u> The statistic $\overline{X}$ is preferred to $\tilde{X}$ because the variance of $\overline{X}$ is smaller than the variance of $\tilde{X}$. Therefore, the estimator $\overline{X}$ is more efficient than the estimator $\tilde{X}$. Results that the estimator $\overline{X}$ is called efficient or optim estimator.

The estimator $\overline{X}$ is an consistent and unbiased estimator (absolute corect estimator) for the unknown parameter m, that is

$$\overline{X} \in \tilde{\mathbf{M}}_n(\hat{\theta}_n) = = \{\hat{\theta}_n \mid E(\hat{\theta}_n) = m, Var(\hat{\theta}_n) \to 0\},$$
$$n \to \infty,$$

where $\tilde{\mathbf{M}}_n(\hat{\theta}_n)$ is the family of all absolute corect estimators for the unknown parameter m and we have the relation

$$Var(\overline{X}) = \frac{\sigma^2}{n} = \min_{\hat{\theta}_n \in \tilde{\mathbf{M}}_n(\hat{\theta}_n)} Var(\hat{\theta}_n).$$

Therefore $\overline{X}$ is the efficient estimator for the unknown parameter $E(X) = m$, that is

$$\hat{\theta}_{n,eff} = \overline{X} = \frac{1}{n}\sum_{k=1}^{n} X_k.$$

<u>Definition 3</u> The efficiency of an inefficient estimate is defined as the ratio of the variances of the efficient and inefficient estimates based on the same number of observations, namely

$$e(\hat{\theta}_{n,ineff}) = \frac{Var(\hat{\theta}_{n,eff})}{Var(\hat{\theta}_{n,ineff})},$$

and, there is

$$0 < e(\hat{\theta}_{n,ineff}) < 1.$$

<u>Definition 4</u> The percentage loss in efficiency by using an inefficient estimate is

$$\overline{e}(\hat{\theta}_{n,ineff}) = 100 \times [1 - e(\hat{\theta}_{n,ineff})].$$

<u>Observation 1</u> For example, in the case of estimators $\tilde{X}$, we have $e(\tilde{X}) \approx \frac{3}{4}$, and $\overline{e}(\tilde{X}) \approx 100 \times [1 - \frac{3}{4}] = 25.$ So, there is 25% loss in efficiency. In other words, by using $\tilde{X}$ as an estimate of m, we have in effect lost a quarter of the sample observations.

CONCLUSIONS

Main results are the proof of the Theorem 1, Conclusion and Observation 1. The entire article including chapter 1: Estimation of the parameters and chapter 2: The efficiency of an estimator consists my own point of view concerning the estimation theory in general and the efficient estimators in particular.

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